Bounds for state-dependent quantum cloning

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Due to the no-cloning theorem, the unknown quantum state can only be cloned approximately or exactly with some probability. There are two types of cloners: universal and state-dependent cloner. The optimal universal cloner has been found and could be viewed as a special state-dependent quantum cloner which has no information about the states. In this paper, we investigate the state-dependent cloning when the state-set contains more than two states. We get some bounds of the global fidelity for these processes. This method is not dependent on the number of the states contained in the state-set. It is also independent of the numbers of copying.

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I. INTRODUCTION

The no-cloning theorem is one of the most important characters of quantum information, which is different from classical information. On the basis of superposition principle, Wootters and Zurek [1] pointed out that it is impossible to find a way to copy an arbitrary unknown state perfectly. They introduced a cloner which is named Wootters-Zurek Copying Machine (W-Z CM). This machine can copy orthogonal state perfectly, but copy the superposition states badly. Since determinately perfect copying is impossible, the approximate cloning is necessary. Bužek and Hillery [2] have first shown that the universal cloner is possible and introduced a copying machine which is called Bužek-Hillery Copying Machine

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(B-H CM). This machine is deterministic and does not need any information about the state to be cloned. It can copy every state equally well. Then it has been proved [3,4] that the B-H CM is the optimal cloning machine for universal cloning, that is, this machine attains the largest local fidelity. There is another kind of cloner which is named state-dependent cloner. It needs some information about the cloning state. There are three types of this kind of cloner: deterministic, probabilistic and hybrid cloners. Probabilistic cloner has been introduced by Duan and Guo [5,6]. They found the states could be cloned perfectly with some probability less than 1,when the states are linearly independent. The deterministic state-dependent cloner was first investigated by Bruß *et al.* [7] and it was solved completely when the state-set contains only two states which have equally *a priori* probability. Then Chefles and Barnett [8] generalized this problem to the two states which have different *a priori* probability and the global fidelity is used to measure of the cloning process instead of the local fidelity. They gave the optimal strategy to make the global fidelity maximal and found this fidelity is larger than the universal cloner. Several months latter, Chefles and Barnett [8] hybridized the former two types of cloners to get the hybridized cloner. So the former two cloners can be viewed as a special case of it.

When the state-set contains only two states, there is a analytic solution of the optimal strategy. Unfortunately, when the number of the states is more than two, there is no analytic solution for this problem. Before the exact solution of 2-state-dependent quantum cloner was found, some scientists had already began to study the bound of these processes. The original work was proposed by Hillery and Bužek [9], they derived a lower bound for the amount of the noise introduced by quantum cloning process. More recently, Rastegin [10,11] gave another lower bound for the noise by a new method. Since solving this problem exactly is impossible when state-set contains more than two states, it is necessary to find the bound of the global fidelity of the quantum cloning process. By the way, when the number of the states is increasing, these states are no longer linearly independent. So the Duan-Guo cloning machine does not work. Even though the bound can not tell us what we can do, it can only tell us what we can not attain. In this paper, some bounds on the multi-state-dependent quantum cloning process are given. We study the three-state-dependent (that is, the state-set contains three states) quantum copying more carefully and generalize the method to the multi-state-dependent cloning process. In Section II, some necessary lemmas are introduced. In Section III, the upper bound of the global fidelity of 3-state-dependent quantum cloning process is given. In Section IV, some upper bounds for n-state-dependent quantum copying process are introduced. The conclusion is given in Section V.

II. NECESSARY PREMISE

Consider a set of n nonorthogonal quantum states $|\psi_i\rangle\,(1 \leq i \leq n)$. If there are M quantum systems, they are prepared in the same unknown quantum state $|\phi\rangle$ which is taken from the given set. The task is to find an optimal process to get $N > M$ identical approximate cloning states from the M initial states $|\phi\rangle$. This process is a symmetric cloning, which can be denoted by $M \to N$. The optimal process means the global fidelity is maximal. The global fidelity is defined as follows:

$$\begin{aligned} F_{MN} &= \sum_{j=1}^{n} \eta_j \left| \left\langle \Psi_j^N \right| \Phi_j \rangle \right|^2 \\ &= \sum_{j=1}^{n} \eta_j \left| \left\langle \Psi_j^N \right| U \left| \Psi_j^M \right\rangle \otimes |0\rangle \right|^2 , \end{aligned} \tag{1}$$

where $|\Phi_j\rangle$ denotes the actual N copies of cloned state of $|\psi_j\rangle$, $\left|\Psi_j^N\right\rangle$ denotes the exact N copies of cloned state of $|\psi_j\rangle$ which is a N-fold tensor and η_j stands for the *a priori* probability of the state $\left|\psi_j^M\right\rangle$, and $|0\rangle$ denotes the $N-M$ blank copies.

The case of the state-set only containing two states has already been solved by Bruß *et al.* [7] and Chefles and Barnett [8]. When they derived the optimal strategy, the following fact is crucial: the optimal outputs $|\Phi_\pm\rangle$ lie in the subspace spanned by the exact clones $\left|\psi_\pm^N\right\rangle$. It is also held when the state-set has more than two states. In fact, we have the following lemma:

Lemma 1. For any state set $S = \{|\psi_1\rangle, |\psi_2\rangle \cdots |\psi_n\rangle\}$ (assume the *a priori* probability of each state are $\frac{1}{n}$) and for the quantum cloning process $M \to N$, the optimal outputs $\{|\Phi_1\rangle, |\Phi_2\rangle \cdots |\Phi_n\rangle\}$ lie in the subspace spanned by the exact clones $\{\left|\psi_1^N\right\rangle, \left|\psi_2^N\right\rangle \cdots \left|\psi_n^N\right\rangle\}$.

Proof.

This proof is following the method introduced by Bruß *et al.* [7].

At the beginning of this proof, we can define a matrix Ξ of the state-set as

$$\Xi = \begin{pmatrix} \langle\psi_1|\,\psi_1\rangle & \langle\psi_1|\,\psi_2\rangle & \cdots & \langle\psi_1|\,\psi_n\rangle \\ \langle\psi_2|\,\psi_1\rangle & \langle\psi_2|\,\psi_2\rangle & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_n|\,\psi_1\rangle & \cdots & \cdots & \langle\psi_n|\,\psi_n\rangle \end{pmatrix}. \tag{2}$$

This matrix is necessary in the following and the definition shows that it is a Hermite matrix.

Suppose that the optimal outputs have the other components which do not lie in the subspace spanned by the exact clones. Then the optimal outputs can be written as

$$\begin{aligned} U\left|\psi_1^M\right\rangle\otimes|0\rangle &= |\Phi_1\rangle = a_{11}\left|\psi_1^N\right\rangle + a_{12}\left|\psi_2^N\right\rangle + \cdots + a_{1n}\left|\psi_n^N\right\rangle + b_1\left|\Gamma_1\right\rangle \\ U\left|\psi_2^M\right\rangle\otimes|0\rangle &= |\Phi_2\rangle = a_{21}\left|\psi_1^N\right\rangle + a_{22}\left|\psi_2^N\right\rangle + \cdots + a_{2n}\left|\psi_n^N\right\rangle + b_2\left|\Gamma_2\right\rangle \\ &\vdots \\ U\left|\psi_n^M\right\rangle\otimes|0\rangle &= |\Phi_n\rangle = a_{n1}\left|\psi_1^N\right\rangle + a_{n2}\left|\psi_2^N\right\rangle + \cdots + a_{nn}\left|\psi_n^N\right\rangle + b_n\left|\Gamma_n\right\rangle, \end{aligned} \tag{3}$$

where the vectors $|\Gamma_1\rangle, |\Gamma_2\rangle \cdots |\Gamma_n\rangle$ are normalized and orthogonal to the subspace spanned by the exact clones, and $|0\rangle$ denotes the $N-M$ blank copies. Since the transformation is unitary, the following constraints must be held.

$$\Omega_{ij}^1 = \mathrm{Re}[\sum_{k,l}\Xi_{kl}^N a_{ik}^* a_{jl} + b_i^* b_j \left\langle\Gamma_i\right|\Gamma_j\rangle - \Xi_{ij}^M] = 0, \tag{4}$$

$$\Omega_{ij}^2 = \mathrm{Im}[\sum_{k,l}\Xi_{kl}^N a_{ik}^* a_{jl} + b_i^* b_j \left\langle\Gamma_i\right|\Gamma_j\rangle - \Xi_{ij}^M] = 0,$$

where $\Xi_{kl}^N = \left\langle\psi_k^N\right|\left.\psi_l^N\right\rangle = (\Xi_{kl})^N$, N denotes the number of copies and M denotes the number of initial identical states. Particularly, when $i=j$, there is the following constraints

$$\Omega_{ii} = \sum_{k,l}\Xi_{kl}^N a_{ik}^* a_{il} + |b_i|^2 - 1 = 0. \tag{5}$$

The global fidelity is

$$\begin{aligned} F_{MN} &= \frac{1}{n}\sum_{i=1}^n\left|\sum_{j=1}^n\Xi_{ij}^N a_{ij}\right|^2 \\ &= \frac{1}{n}\sum_{i=1}^n[\sum_{k,l}(\Xi_{ik}^N\Xi_{il}^{*N} - \Xi_{lk}^N)a_{il}^* a_{ik} + \sum_{k,l}\Xi_{lk}^N a_{il}^* a_{ik}] \\ &= \frac{1}{n}\sum_{i=1}^n(1-|b_i|^2) + \frac{1}{n}\sum_{i=1}^n[\sum_{k,l}(\Xi_{ik}^N\Xi_{il}^{*N} - \Xi_{lk}^N)a_{il}^* a_{ik}. \end{aligned} \tag{6}$$

The constraints (5) have already been used. Now we can use the Lagrange multipliers for the other constraints and get these equations

$$\frac{\partial F_{MN}}{\partial|a_{ij}|} + \sum_{k,l,\sigma}\lambda_{kl}^\sigma\frac{\partial\Omega_{kl}^\sigma}{\partial|a_{ij}|} = 0, \tag{7}$$

$$\frac{\partial F_{MN}}{\partial |b_i|}+\sum_{k,l,\sigma}\lambda_{kl}^{\sigma}\frac{\partial \Omega_{kl}^{\sigma}}{\partial |b_i|}=0, \tag{8}$$

$$\frac{\partial F_{MN}}{\partial |\langle \Gamma_i\, |\Gamma_j\rangle|}+\sum_{k,l,\sigma}\lambda_{kl}^{\sigma}\frac{\partial \Omega_{kl}^{\sigma}}{\partial |\langle \Gamma_i\, |\Gamma_j\rangle|}=0 \tag{9}$$

etc, Where Ω_{kl}^{σ} denotes the constraints, and the Lagrange multipliers are λ_{kl}^{σ}. Since the constraint when $k=l$ has been used before, the index k and l in all of the equations must satisfy $k\neq l$. We suppose that $b_i=|b_i|\, e^{i\delta_i}$ and $\langle \Gamma_k|\, \Gamma_i\rangle=|\langle \Gamma_k|\, \Gamma_i\rangle|\, e^{i\delta_{ki}}$. Due to the equality of Ω_{kl}^{σ} and Ω_{lk}^{σ}, we can only consider one of them. Then Eq. (8) and Eq. (9) can be written as

$$-\frac{2}{n}\,|b_i|+\sum_{k=1}^{n}\lambda_{ki}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_k)}\,|b_k|\,\langle \Gamma_k|\, \Gamma_i\rangle)+\sum_{k=1}^{n}\lambda_{ki}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_k)}\,|b_k|\,\langle \Gamma_k|\, \Gamma_i\rangle)=0, \tag{10}$$

$$\lambda_{ji}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_j+\delta_{ji})}\,|b_i|\,|b_j|)+\lambda_{ji}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_j+\delta_{ji})}\,|b_i|\,|b_j|)=0. \tag{11}$$

Let us multiply Eq. (10) by $|b_i|$, we get

$$-\frac{2}{n}\,|b_i|^2+\sum_{k=1}^{n}\lambda_{ki}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_k)}\,|b_k|\,|b_i|\,\langle \Gamma_k|\, \Gamma_i\rangle)+\sum_{k=1}^{n}\lambda_{ki}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_k)}\,|b_k|\,|b_i|\,\langle \Gamma_k|\, \Gamma_i\rangle)=0. \tag{12}$$

After multiplying Eq. (11) by $|\langle \Gamma_j|\, \Gamma_i\rangle|$, we find

$$\lambda_{ji}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_j+\delta_{ji})}\,|b_i|\,|b_j|\,|\langle \Gamma_j|\, \Gamma_i\rangle|)+\lambda_{ji}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_j+\delta_{ji})}\,|b_i|\,|b_j|\,|\langle \Gamma_j|\, \Gamma_i\rangle|)=0,$$

that is,

$$\lambda_{ji}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_j)}\,|b_i|\,|b_j|\,\langle \Gamma_j|\, \Gamma_i\rangle)+\lambda_{ji}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_j)}\,|b_i|\,|b_j|\,\langle \Gamma_j|\, \Gamma_i\rangle)=0.$$

Then we sum them over the subscript j from1 to n and get

$$\sum_{j=1}^{n}[\lambda_{ji}^{1}\,\mathrm{Re}(e^{i(\delta_i-\delta_j)}\,|b_i|\,|b_j|\,\langle \Gamma_j|\, \Gamma_i\rangle)+\lambda_{ji}^{2}\,\mathrm{Im}(e^{i(\delta_i-\delta_j)}\,|b_i|\,|b_j|\,\langle \Gamma_j|\, \Gamma_i\rangle)]=0. \tag{13}$$

Substituting Eq.(13) into Eq. (12) and changing the subscript j to k, we can find that $|b_i|^2=0$, that is, $|b_i|=0$. This is the end of the proof.

Note that the lemma is also held when the *a priori* probability is not equal for all of the states. The proof is the same as before. What we need to do is to change the *a priori* probability $\frac{1}{n}$ by the new *a priori* probability $\eta_{i.}$

Now we consider the global fidelity formula Eq. (6). For convenience, let $a_{ij} = |a_{ij}| e^{i\sigma_{ij}}$. If we assume that the elements of matrix Ξ are real, in order to make the global fidelity maximal, factors $e^{i\sigma_{ij}}$ and $e^{i\sigma_{ik}}$ must be the same. So we can write the Eq. (3.1) in a new form.

$$\begin{aligned}
|\Phi_1\rangle &= e^{i\sigma_1}(a_{11}\left|\psi_1^N\right\rangle + a_{12}\left|\psi_2^N\right\rangle + \cdots + a_{1n}\left|\psi_n^N\right\rangle) \\
|\Phi_2\rangle &= e^{i\sigma_2}(a_{21}\left|\psi_1^N\right\rangle + a_{22}\left|\psi_2^N\right\rangle + \cdots + a_{2n}\left|\psi_n^N\right\rangle) \\
&\vdots \\
|\Phi_n\rangle &= e^{i\sigma_n}(a_{n1}\left|\psi_1^N\right\rangle + a_{n2}\left|\psi_2^N\right\rangle + \cdots + a_{nn}\left|\psi_n^N\right\rangle),
\end{aligned} \tag{14}$$

where a_{ij} are real numbers. So it is sufficient to consider the real number coefficients to find the maximum of the global fidelity. In the next section we only study the global fidelity in this sense.

III. SOME BOUNDS FOR STATE-DEPENDENT CLONING WHEN STATE-SET CONTAINS THREE STATES

Now we consider the situation that the state set contains three states $\{\psi_1, \psi_2, \psi_3\}$. We assume that the three states are taken from state-set $\{\sin\theta\,|1\rangle + \cos\theta\,|0\rangle\,,\ 0 \le \theta \le \frac{\pi}{2}\}$. The elements of the matrix Ξ are naturally real. We consider the quantum cloning process $M \to N$. The quantum state in the space spanned by $\{\left|\psi_1^N\right\rangle, \left|\psi_2^N\right\rangle, \left|\psi_3^N\right\rangle\}$ is a point on the complex spherical surface (in general, the states $\left|\psi_1^N\right\rangle, \left|\psi_2^N\right\rangle, \left|\psi_3^N\right\rangle$ are linearly independent and can span a 3-dimensional space). With the reason pointed out before, when considering the optimal cloning strategy, we can only consider the states which have real coefficients, and these states span the spherical surface S^2. Finding the optimal clone is equal to finding three points on the S^2 which make the distances between them and the idea copies minimal. This situation is described in Fig. 1. In this figure, the edge of the outer triangle a corresponds the angle between $|\psi_2\rangle^N$ and $|\psi_3\rangle^N$, that is, $\cos a = (\langle\psi_2\,|\psi_3\rangle)^N$. The edge of the inner triangle a' corresponds the angle between $|\psi_2\rangle^M$ and $|\psi_3\rangle^M$, that is, $\cos a' = (\langle\psi_2\,|\psi_3\rangle)^M$. And so on.

In order to get the optimal approximate of the global fidelity, we must give some characters of the spherical surface S^2.

Lemma 2. For the triangle on the spherical surface, there is a fundamental formula [12]

$$\cos a = \cos b \cos c + \sin a \sin b \cos \alpha,$$

where a, b, c are the length of the three edges of this triangle and α is the angle between edge b and c.

From lemma 2, the following equations can be obtained from Fig. 1 (We suppose $\alpha = \angle BAC$, $\theta = \angle BAA^{'}$, $\Phi_1 = \angle AA'B$, $\varphi = \angle AA^{'}C$, $\Phi = \angle AA^{'}B^{'}$,$\beta = \angle C^{'}A^{'}B^{'}$and $c + l \leq \frac{\pi}{2}, b + l \leq \frac{\pi}{2}$)

$$\cos m = \cos c \cos l + \sin c \sin l \cos \theta, \tag{15-1}$$

$$\frac{\sin c}{\sin \Phi_1} = \frac{\sin m}{\sin \theta} \tag{15-2}$$

and

$$\cos n = \cos b \cos l + \sin b \sin l \cos(\alpha - \theta), \tag{16-1}$$

$$\frac{\sin b}{\sin \varphi} = \frac{\sin n}{\sin (\alpha - \theta)}. \tag{16-2}$$

Now we can calculate out that

$$\begin{aligned} \cos l_2 &= \cos m \cos c^{'} + \sin m \sin c^{'} \cos(\Phi - \Phi_1) \\ &\leq \cos m \cos c^{'} + \sin m \sin c^{'} \\ &\leq \cos m \cos c^{'} + \sin(c + l) \sin c^{'}. \end{aligned} \tag{17}$$

We have already used the condition $c+l \leq \frac{\pi}{2}$, and inequalities $m \leq c+l$ (it is proven in the following), that is, $\sin m \leq \sin(c+l)$. Substituting Eq. (14-1) into Eq. (16) and rearranging it in order of $\cos l$ and $\sin l$, we get

$$\cos l_2 \leq \cos(c - c^{'}) \cos l + (\sin c \cos c^{'} \cos \theta + \cos c \sin c^{'}) \sin l. \tag{18}$$

For the same reason, we can get

$$\begin{aligned} \cos l_3 &= \cos n \cos b^{'} + \sin n \sin b^{'} \cos(2\pi - \Phi - \beta - \varphi) \\ &\leq \cos n \cos b^{'} + \sin n \sin b^{'} \\ &\leq \cos n \cos b^{'} + \sin(b + l) \sin b^{'}. \end{aligned} \tag{19}$$

Now inserting Eq.(15-1) into this formula and using the fact that $\sin \theta \leq \sin \alpha$, we get

$$\cos l_3 \leq [\cos b \cos l + \sin b \sin l \cos(\alpha - \theta)] \cos b' + \sin(b + l) \sin b' \qquad (20)$$
$$\leq \cos(b - b') \cos l + [\sin b \cos b' \cos\alpha \cos\theta + (\sin b \cos b' \sin^2\alpha + \cos b \sin b')] \sin l$$

From the definition of the global fidelity, we insert Eq. (14) and Eq. (15) into the fidelity formula to get

$$\begin{aligned} F_g &= \frac{1}{3}(\cos^2 l + \cos^2 l_2 + \cos^2 l_3) \\ &\leq \frac{1}{3}(\cos^2 l + [\cos(c - c') \cos l + (\sin c \cos c' \cos\theta + \cos c \sin c') \sin l]^2 \\ &\quad +[\cos(b - b') \cos l + [\sin b \cos b' \cos\alpha \cos\theta + (\sin b \cos b' \sin^2\alpha + \cos b \sin b')] \sin l]^2). \end{aligned} \qquad (21)$$

So the upper bound of the fidelity must be less than the maximum of the right hand of Eq.(16). Before getting the result, let $A_1 = \cos(c - c')$, $A_2 = \sin c \sin c'$, $A_3 = \cos c \sin c'$; $B_1 = \cos(b - b')$, $B_2 = \sin b \cos b' \cos\alpha$, $B_3 = \sin b \cos b' \sin^2\alpha + \cos b \sin b'$. Then the maximum of the right hand of Eq. (16) is

$$\frac{1}{3}(\cos^2 l + [A_1 \cos l + (A_2 + A_3) \sin l]^2 + [B_1 \cos l + (B_2 + B_3) \sin l]^2), \qquad (22)$$

where l satisfies the condition

$$tg(2l) = \frac{2[A_1(A_2 + A_3) + B_1(B_2 + B_3)}{1 + A_1^2 + B_2^2 - (A_2 + A_3)^2 - (B_2 + B_3)^2}. \qquad (23)$$

It can be seen from these formulas that they are symmetric for A and B.

We can find another interesting thing from this spherical surface S^2. We can attain all of the results of Rastegin [10,11] succinctly and directly from Cauchy Lemma , which has obvious geometric meaning. The Cauchy Lemma on the S^2 is very important and useful. It is given out as the following.

Lemma 3. (*Cauchy* Lemma) There are two polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$. If the lengths of the edges in these two polygons satisfy $A_1A_2 = B_1B_2$, $A_2A_3 = B_2B_3$, $\cdots$, $A_{n-1}A_n = B_{n-1}B_n$ and the angles satisfy $A_2 \leq B_2$, $A_3 \leq B_3$, $\cdots$, $A_{n-1} \leq B_{n-1}$. Then $A_1A_n \leq B_1B_n$.

This lemma looks very simple, but its proof is rather difficult, the proof of this lemma can be found in [12]. This Lemma is useful to get some inequality. When $n = 3$, we get the familiar inequality for spherical surface triangle

$$A_1A_3 - A_3A_2 \leq A_1A_2 \leq A_1A_3 + A_3A_2.$$

The left part of this formula is the edge of a special triangle $A_1A_3A_2$ whose angle A_3 is 0. While any angle of a triangle must be not less than 0, the left inequality is held. For the same reason, the right inequality is also hold. This inequality is just the same as $\cos\delta_{\Phi\Psi} \leq \cos(\delta_{\Phi\Upsilon} - \delta_{\Upsilon\Psi})$ which was introduced by Rastegin [10]. When $n = 4$ we can get some useful inequality (Fig. 1)

$$l + c^{'} + l_2 \geq c, \; l + b^{'} + l_3 \geq b, \; l_2 + a^{'} + l_3 \geq a. \tag{24}$$

This inequality is the same as $\cos(\delta_{\Gamma\Lambda} + \delta_{\Gamma\Xi} + \delta_{\Lambda\Sigma}) \leq \cos\delta_{\Xi\Sigma} \left(\delta_{\Gamma\Lambda} + \delta_{\Gamma\Xi} + \delta_{\Lambda\Sigma} \leq \frac{\pi}{2}\right)$ in the Rastegin's paper [10]. Using Cauchy lemma, we can make the condition weaken to $\delta_{\Gamma\Lambda} + \delta_{\Gamma\Xi} \leq \frac{\pi}{2}$, and get the useful new inequality $\cos(\delta_{\Gamma\Lambda} + \delta_{\Gamma\Xi}) \leq \cos(\delta_{\Xi\Sigma} - \delta_{\Lambda\Sigma})$. From these new inequalities we can get a new upper bound of the three states global fidelity

$$\begin{aligned} F_g &= \frac{1}{3}(\cos^2 l + \cos^2 l_2 + \cos^2 l_3) \\ &\leq \frac{1}{6}(3 + \cos(l + l_1) + \cos(l_1 + l_2) + \cos(l_2 + l_3)) \\ &\leq \frac{1}{6}(3 + \cos(a - a^{'}) + \cos(b - b^{'}) + \cos(c - c^{'})). \end{aligned} \tag{25}$$

This bound of the global fidelity is symmetric to the three edges of the triangle, that is, symmetric to the three states. This formula is more simple than Eq.(17). The equal sign is held when the states in the state-set are orthogonal.

IV. SOME BOUNDS OF STATE-DEPENDENT CLONING WHEN STATE-SET CONTAINS N STATES

When the state-set contains more than three states, they can span a space more than three dimensions. In this situation, the actual quantum cloned states are points on a spherical surface more than 2-dimension. The method which we used to get Eq. (17) on S^2 is not available. Fortunately, the inequality (19) for any four states on the same spherical surface is still correct. So we can use this inequality to get some upper bounds of the global fidelity of multi-state-dependent quantum cloning.

Assume the state-set contains n states $\{\psi_1, \psi_2, \cdots, \psi_n\}$. And these states are taken from $\{\sin\theta\,|1\rangle + \cos\theta\,|0\rangle\,, 0 \leq \theta \leq \frac{\pi}{2}\}$. At first, we divide the state-set $\{\psi_1^N, \psi_2^N, \cdots, \psi_n^N\}$ into several groups $\{\{\psi_1^N, \psi_2^N, \cdots, \psi_i^N\}$, $\{\psi_{i+1}^N, \psi_{i+2}^N, \cdots, \psi_j^N\}$, $\cdots$, $\{\psi_{k+1}^N, \psi_{k+2}^N, \cdots, \psi_n^N\}\}$, and every group constitutes a convex polygon on the same S^2 (We can obtain this result by the following step. At first, we can take any three states from the state-set and they must constitute the spherical surface S^2. Then put all of the vectors which are linearly dependent on the three states and make the points of all of these vectors constitute a convex polygon. Do the same operation to the rest vectors until the number of the vectors is less than 3. At this situation, we take some vectors from the group which has more than three vectors to get a new spherical surface S^2). On the spherical surface S^2, we can use the Cauchy lemma to get the inequality (19) and insert them into the formula of the global fidelity.

$$\begin{aligned}
F_g &= \frac{1}{n}[\sum_{p=1}^{i}\cos^2 l_p + \sum_{q=i+1}^{j}\cos^2 l_q + \cdots + \sum_{r=k+1}^{n}\cos^2 l_{,r}] \qquad (26)\\
&\leq \frac{1}{2n}[(i + \sum_{p=1}^{i}\cos(a_{p+1,p} - a^{'}_{p+1,p})) + \sum_{q=i+1}^{j}((j-i) + \cos(a_{q+1,q} - a^{'}_{q+1,q})) + \cdots \\
&\quad + \sum_{r=k+1}^{n}((n-k) + \cos(a_{r+1,r} - a^{'}_{r+1,r})] \\
&= \frac{1}{2} + \frac{1}{2n}[\sum_{p=1}^{i}\cos(a_{p+1,p} - a^{'}_{p+1,p}) + \sum_{q=i+1}^{j}\cos(a_{q+1,q} - a^{'}_{q+1,q}) + \cdots \\
&\quad + \sum_{r=k+1}^{n}\cos(a_{r+1,r} - a^{'}_{r+1,r})],
\end{aligned}$$

where $\cos l_i = \langle\Phi_i \left|\psi_i^N\right\rangle$ and $\cos a_{i+1,i} = \left\langle\psi_{i+1}^N \left|\psi_i^N\right\rangle\right.$, $\cos a^{'}_{i+1,i} = \left\langle\psi_{i+1}^M \left|\psi_i^M\right\rangle\right.$ and $1, 2, \cdots i;\ i+1, i+2, \cdots, j;\ \ \cdots;\ k+1, k+2, \cdots n$ are vertexes of a convex polygon on S^2 respectively.

It can be seen from the deriving process that the result is dependent on the partition of the states and the choice of the loops. It is not good enough for us since it is not uniquely determined by the state-set. Since the Eq. (19) is correct for every four vectors, we can get a more symmetric result. We can average all of the possible divided sets and get

$$F_g \leq \frac{1}{2} + \frac{1}{2n(n-1)}[\sum_{k\neq j=1}^{n}\cos(a_{j,k} - a^{'}_{j,k})]. \qquad (27)$$

Even though this inequality is the most symmetry for every state-set, it is not the tightest bound which we can get by this method for the quantum cloning process. There are some methods to refine the bound. First, we construct the $n \times n$ matrix M whose elements are $a_{j,k} - a^{'}_{j,k}$, So the matrix is

$$M = \begin{pmatrix} a_{1,1} - a^{'}_{1,1} & a_{1,2} - a^{'}_{1,2} & \cdots & a_{1,n} - a^{'}_{1,n} \\ a_{2,1} - a^{'}_{2,1} & a_{2,2} - a^{'}_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a^{'}_{n,1} & \cdots & \cdots & a_{n,n} - a^{'}_{n,n} \end{pmatrix}. \tag{28}$$

It should be pointed out that the diagonal elements of this matrix are zero. The first step of our refining process is to find the maximal element $M_{mn}(m > n)$ in the upper triangle of the matrix which is denoted by e_1. Then we replace the row and the column in which the element e_1 lies by zeroes. At the end of the first step we get an element and a new $n \times n$ matrix. (If there are i maximal elements which are denoted by $M_{l_1 k_1}$, $M_{l_2 k_2} \cdots M_{l_i k_i}$ in the upper triangle of the matrix M. Then we denote the secondary maximum element of upper triangle elements within the rows l_j and columns k_j as $M_{l^{'}_j k^{'}_j}$. Thus we denote the minimal element $M_{l^{'}_{j0} k^{'}_{j0}}$ of $M_{l^{'}_j k^{'}_j}$ as e_1). We can iterate the operation to get the elements e_3 ,e_4 ,$\cdots$, $e_{[\frac{n+1}{2}]}$($[x]$ is the integer part of x). Then a more stringent bound can be attained with our method

$$F_g \leq \frac{1}{2} + \frac{1}{2[\frac{n+1}{2}]}[\sum_{j=1}^{[\frac{n+1}{2}]} \cos e_j]. \tag{29}$$

V. CONCLUSION

In the practical quantum information processes the state-dependent cloning is more important than the universal cloning. In fact, we can view the universal cloning as the lower bound of the state-dependent cloning. If we have no information about the state-set, the optimal strategy we can chose is the universal cloning. But if we know something about the state-set, that is, we have some information about the state, we can find the strategy no worse than the universal cloning. In fact, we always have some information about the states in the practical information processes. So it is very important to find the strategy of the state-dependent cloning which is better than universal clone. At the same time we want to know *how much* we can improve the quantum cloning process when we know something about the state-set. Unfortunately, it is too difficult to solve this problem completely. We can only find some bounds for this process and partly answer this question.

There are close relations between state-dependent cloning and eavesdropping in quantum cryptography. Cloning is a method for eavesdropper to eavesdropping (but it is not necessarily the optimal one). When the state-set only contains two states, the situation has already been completely discussed by Bruß *et al.* [7]. Because of the relationship between the quantum cloning and eavesdropping, the bound of the multi-state-dependent quantum cloning can be considered as a bound for the eavesdropping when the eavesdropper use the cloning strategy to get the information of the communication process.

VI. ACKNOWLEDGMENT

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[13] **Figure caption**

Figure 1. A, B, C are the vertexes of the idea copies of $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ and A', B', C' are the vertexes of the virtual copies of $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$. They are on the same spheral surface. a, b, c, a', b', c' and m, n, l, l_2, l_3 are arcs on this spheral surface. We suppose $\angle BAA' = \theta$, $\angle AA'B = \Phi_1$, $\angle AA'C = \varphi$, $\angle AA'B' = \Phi$, $\angle BAC = \alpha, \beta = \angle C'A'B'$.

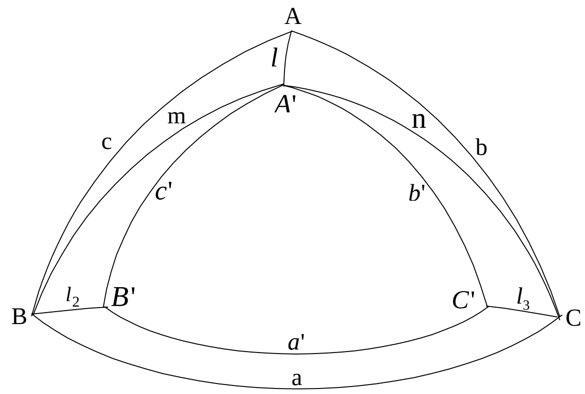

Figure 1